

# Shape Invariant Rational Extensions And Potentials Related to Exceptional Polynomials

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## Abstract

In this paper, we show that an attempt to construct shape invariant extensions of a known shape invariant potential leads to, apart from a shift by a constant, the well known technique of isospectral shift deformation. Using this, we construct infinite sets of generalized potentials with  $X_m$  exceptional polynomials as solutions. These potentials are rational extensions of the existing shape invariant potentials. The method is elucidated using the radial oscillator and the trigonometric Pöschl-Teller potentials. For the case of radial oscillator, in addition to the known rational extensions, we construct two infinite sets of rational extensions, which seem to be less studied. For one of the potential, we show that its solutions involve a third type of exceptional Laguerre polynomials. Explicit expressions of this generalized infinite set of potentials and the corresponding solutions are presented. For the trigonometric Pöschl-Teller potential, our analysis points to the possibility of several rational extensions beyond those known in literature.

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# 1 Introduction

The search for new exactly solvable (ES) models gained momentum with the discovery of the exceptional orthogonal polynomials (EOPs) [1],[2]. New potentials with the  $X_1$  EOPs as solutions were constructed in [3],[4]. They turned out to be the rational extensions of the well studied one dimensional bound state problems namely the radial oscillator and the Darboux Pöschl-Teller (DPT) potentials [5]. A spate of papers followed, where generalized rational extensions of many one dimensional shape invariant potentials (SIPs) were presented [6]- [17]. Different methods like the Darboux-Crum method [10], [11], finite difference Backlund algorithm [12]-[14], prepotential method [15] and several others [16], [17] were employed. Various groups explored the mathematical properties of the new polynomials concurrently [18] - [20].

Many studies showed that the traditional SIPs and their rational extensions are isospectral and share a supersymmetric partnership [3], [16], [17]. In recent papers, Grandati *et.al.*, [12], [13] constructed rational extensions of SIPs using regularized excited states Riccati-Schrödinger functions as superpotentials. The finite difference Bäcklund transformation was combined with the regularization scheme to generate infinite sets of rational extensions of the isotonic oscillator [13]. Dutta, Roy and coworkers [16], [21] and references therein, constructed new conditionally exactly solvable (CES) models using supersymmetric techniques. Among these a particular construction gave potentials related to exceptional polynomials [21].

Shape invariance (SI) property, discovered in [22], has been central to the discussion of the exact solutions of potential problems in supersymmetric quantum mechanics (SUSYQM) [5], [23], [24]. We make this property the focus of our study in this paper. Here we formulate a sufficient condition for SI using the quantum Hamilton Jacobi (QHJ) formalism [25] - [30]. This condition brings out the fact that SI is a property of a potential and does not require knowledge or a reference to a SUSY partner potential or the superpotential. This requirement makes it possible to analyze the SI property and use it to obtain deformed potentials which are shape invariant by construction. This will be illustrated using the radial oscillator and the Darboux Pöschl-Teller (DPT) potentials. In an earlier paper [31], it has been shown that all previously known SIPs, as listed in [5], can be derived in an extremely simple, straightforward method using the sufficient condition for SI. In this paper we complete this program by constructing rational SIPs of radial oscillator leading to potentials related to the  $L1$  and  $L2$  exceptional polynomials [8], [10], [11], [13], [14]. In addition, we also find one more generalized set of potentials with polynomial solutions, which may be classified as type III or  $L3$  exceptional polynomials [8], [13]. The expressions for these generalized potentials and their polynomial solutions are presented here for the first time. For the DPT potential, we are led to many rational extensions of which some coincide with the known extensions in the literature [3], [8], [10].

Our scheme allows us to construct a hierarchy of generalized ES rational potentials indexed by the hierarchy index  $m$ , ( $m = 0, 1, 2 \dots$ ). Each value of  $m$  gives a new rational extension of the original SIP and the solutions are in terms of the  $X_m$  EOPs. For  $m = 0$ , the generalized potentials reduce to the original SIPs. In the scheme presented in this paper, the superpotentials of the original SIP, corresponding to both exact and broken SUSY, play a central role. The interplay of ideas and the use of tools available in the QHJ formalism and SUSYQM makes our scheme very simple, elegant and transparent.

The knowledge of the singularity structure of the quantum momentum function (QMF) within the QHJ formalism, allows one to construct all possible superpotentials associated with a given supersymmetric potential. Next for each superpotential, say  $W(x)$ , we find extended superpotential,  $\widetilde{W}(x)$ , such that the corresponding potentials  $\widetilde{V}^{(+)}(x)$  is shifted w.r.t.  $V^{(+)}(x)$  by a constant. Each of the resulting potentials,  $\widetilde{V}^{(-)}(x)$ , associated with  $\widetilde{W}(x)$ , is either a singular or a regular rational extension of  $V^{(-)}(x)$ . This method has been earlier used in [16] to obtain CES rational potentials with EOPs as solutions. The superpotentials associated with nonnormalizable solutions of  $V^{(-)}(x)$  lead to the regular extensions and these potentials have eigenfunctions in terms of the  $X_m$  exceptional polynomials as solutions. The potentials associated with the normalizable wave functions result in new sets of generalized potentials which are singular or regular depending on the potential parameters. Some of these potentials are discussed in [8], [13].

In Sec. 2 we first recall some important features of the QHJ formalism of quantum mechanics

and reformulate the SI requirement, crucial to our discussion. In Sec.3, we show that this requirement leads to the isospectral shift deformation of the SIPs known from the early days of SUSYQM. Sec.4 has a detailed discussion of rational extensions of the radial oscillator. We also present the expressions for an infinite set of rational potentials, whose solutions are in terms of the new set of polynomials, the  $L_3$  category of exceptional Laguerre polynomials. The case of trigonometric DPT, leading to the exceptional Jacobi polynomials is taken up in Sec.5. Finally we present our conclusions in Sec.6.

## 2 The QHJ and SUSYQM formalisms

In the QHJ formalism [25] - [28], the starting point is the Riccati equation

$$p^2(x, E) - i\hbar \frac{dp(x, E)}{dx} = 2m(E - V(x)), \quad (1)$$

obtained by making the substitution  $\psi(x, E) = \exp(iS(x, E)/\hbar)$  in the Schrödinger equation. Here  $p(x, E)$ , the quantum momentum function (QMF), is identified with  $\frac{dS(x, E)}{dx}$ . For convenience, we will use units so that  $\hbar = 1, 2m = 1$  and in order to work with real equations, we define  $Q(x, E)$  by  $p(x, E) \equiv iQ(x, E)$ . The QHJ equation for  $Q(x, E)$  assumes the form

$$Q^2(x, E) - \frac{dQ(x, E)}{dx} = V(x) - E. \quad (2)$$

$Q(x, E)$  is simply the negative of the logarithmic derivative of solution  $\psi(x, E)$  of the Schrödinger equation

$$Q(x, E) = -\frac{d}{dx} \log \psi(x, E) = -\frac{1}{\psi(x, E)} \frac{d\psi(x, E)}{dx}. \quad (3)$$

The QHJ approach essentially makes use of the singularities of the function  $Q(x, E)$  in the complex  $x$  plane. A crucial step in solving for the QMF,  $Q(x, E)$ , is to write it as a sum of two parts,

$$Q(x, E) = Q_f + Q_m \quad (4)$$

where  $Q_f$  contains the fixed singularities and its form can be written down by inspection of different terms in the QHJ equation. The second part,  $Q_m$ , contains the singular part having the moving singularities which depend on the boundary conditions to be imposed. In all known cases of ES potentials [27], [28] including the new rational potentials with EOP solutions [29] and the quasi-exactly solvable potentials [30], when  $E$  is taken to be one of the energy eigenvalues,  $Q_m$  turns out to have a *finite number of simple poles with residue -1*. It is easy to see from (3) that the moving poles of  $Q(x)$  are located at the nodes and at other complex zeros of the wave function.

The superpotential  $w(x)$ , which in SUSYQM corresponds to the negative of logarithmic derivative of a solution of the Schrödinger equation, plays an important role of defining the partner potential and formulating the SI condition [22]. Usually, the solution chosen has no nodes but may, or may not, be normalizable, the two cases being referred to as cases of exact and broken SUSY respectively. Thus  $w(x)$  happens to be just one of the solutions  $Q(x, E)$  of the QHJ equation, for certain energy, for which moving poles are absent and  $Q_m$  in (4) is replaced by a constant. The SUSY partner potentials  $V^\pm(x)$  are defined by

$$V^{(\pm)}(x) = w(x)^2 \pm w'(x) \quad (5)$$

and a potential  $V(x)$  is defined to be shape invariant if both the partner potentials have the same form as  $V(x)$  apart from an overall constant. Since several superpotentials are possible, hence the definition of partners, and the property of SI itself appears to depend on the choice of the superpotential. It turns out to be extremely useful to formulate SI as a property which does not require the use of the superpotential, and which makes it transparent as a property of the potential. As we will see later, this fact assumes importance in the context of rational extensions related to the exceptional polynomials.

We now reformulate the SI requirement on a potential  $V(x)$ . Let  $Q(x, \sigma)$  be a solution of the QHJ equation,

$$Q^2(x, \sigma) - \frac{dQ(x, \sigma)}{dx} = V(x) - E(\sigma), \quad (6)$$

where the dependence of  $Q(x, \sigma)$  on the potential parameters has been explicitly shown through constants  $\sigma$ , which are functions of the parameters appearing in the potential. If we introduce two functions  $w_1(x), w_2(x)$

$$w_1(x) = Q(x, \lambda), \quad w_2(x) = Q(x, \mu). \quad (7)$$

corresponding to two different values  $\sigma = \lambda, \mu$ , then the two potentials, defined by

$$V_1(x) = w_1^2(x) - w_1'(x), \quad V_2(x) = w_2^2(x) - w_2'(x), \quad (8)$$

obviously will have the same shape. If we further require that there exists a map

$$\tau : \lambda \rightarrow \mu = \tau(\lambda) \quad (9)$$

such that

$$Q(x, \lambda) + Q(x, \mu) = 0, \quad (10)$$

then  $w_2(x) = -w_1(x)$  and the two potentials  $V_{1,2}(x)$  will be SUSY partners.

*To summarize, given a potential  $V(x)$ , the twin requirements of existence of solutions  $Q(x, \lambda)$ ,  $Q(x, \mu)$  of QHJE and of a map  $\tau$  such that*

$$Q(x, \mu) = -Q(x, \tau(\lambda)) \quad (11)$$

*are seen to be the sufficient conditions for SI of the potential  $V(x)$ .*

### 3 Shape Invariant Rational Extensions

Given a superpotential  $w(x, \lambda)$ , the SI requirement means the potential  $V^{(+)}(x, \lambda) = w^2(x, \lambda) + w'(x, \lambda)$  be equal to  $V^{(-)}(x, \mu) \equiv w^2(x, \mu) - w'(x, \mu)$ , apart from an additive constant, after the set of potential parameters,  $\lambda$ , is redefined. This translates into the following requirement on the superpotential as

$$w^2(x, \lambda) + w'(x, \lambda) = w^2(x, \mu) - w'(x, \mu) + \text{constant}. \quad (12)$$

for some  $\lambda, \mu$  and a constant which may depend on  $\lambda, \mu$ .

Let us now assume that a particular superpotential  $w_0(x, \lambda)$  has been found which gives rise to a SIP  $V(x, \lambda)$ , where  $\lambda$  denotes constants which are functions of the potential parameters. We now seek to construct a shape invariant extension of  $w_0(x, \lambda)$  by constructing an extended superpotential

$$\tilde{w}(x, \lambda) = w_0(x, \lambda) + \phi(x, \lambda), \quad (13)$$

where  $\phi(x, \lambda)$  is an unknown function and is to be determined from the requirements of SI.

Using the fact that both  $\tilde{w}(x)$  and  $w_0(x)$  for some  $\mu, \lambda$ , satisfy the SI requirement (12), we get

$$\phi(x, \lambda)^2 + 2w_0(x, \lambda)\phi(x, \lambda) + \phi'(x, \lambda) \quad (14)$$

$$= \phi(x, \mu)^2 + 2w_0(x, \mu)\phi(x, \mu) - \phi'(x, \mu) + K, \quad (15)$$

where  $K$  is a constant. For SIPs of interest in this paper, and defined by  $w_0(x, \lambda)$ , there exists a map  $\tau : \lambda \rightarrow \mu = \tau(\lambda)$  such that

$$w_0(x, \mu) = -w_0(x, \tau(\lambda)). \quad (16)$$

We also want the extended potential to be shape invariant and hence we demand that

$$\phi(x, \lambda) \rightarrow -\phi(x, \tau(\lambda)). \quad (17)$$

Thus we can rewrite (15) as

$$\phi(x, \lambda)^2 + 2w_0(x, \lambda)\phi(x, \lambda) + \phi'(x, \lambda) \quad (18)$$

$$= \phi(x, \tau(\lambda))^2 + 2w_0(x, \tau(\lambda))\phi(x, \tau(\lambda)) + \phi'(x, \tau(\lambda)) + K. \quad (19)$$

The structure of the above equation suggests using the ansatz of equating both sides of the above equation to a constant. Therefore, we look for solutions for  $\phi(x, \lambda)$  satisfying the equation

$$\phi(x, \lambda)^2 + 2w_0(x, \lambda)\phi(x, \lambda) + \phi'(x, \lambda) = R_1, \quad (20)$$

where  $R_1$  is a constant depending on the potential parameters  $\lambda$ . The above condition, (20), is simply the relation

$$\tilde{V}^{(+)}(x) = V^{(+)}(x) + R_1, \quad (21)$$

where the notation  $\tilde{V}^{(\pm)}(x)$  and  $V^{(\pm)}(x)$  denote the SUSY partner potentials constructed from  $\tilde{w}(x)$  and  $w_0(x)$  respectively. It may be remarked that we have recovered the property that has been used in [16] to construct new conditionally exactly solvable models with solutions related to the exceptional polynomials. For the special choice  $R_1 = 0$ , this requirement is same as that used in construction of strictly isospectral deformations of a given potential [32], [33],

$$V^{(-)}(x) = w_0^2(x, \lambda) - w_0'(x, \lambda). \quad (22)$$

In general the constant  $R_1$  in (21) can be non zero and the above process of defining  $\tilde{V}^{(-)}(x)$  by means of

$$\tilde{V}^{(-)}(x) = \tilde{w}^2(x, \lambda) - \tilde{w}'(x, \lambda) \quad (23)$$

will be called the *isospectral shift deformation* of potential  $V^{(-)}(x)$ .

The equation for  $\phi(x, \lambda)$  is the Riccati equation and can be linearised by introducing  $u(x, \lambda)$  defined by

$$\phi(x, \lambda) = \frac{1}{u(x, \lambda)} \frac{du(x, \lambda)}{dx}. \quad (24)$$

The unknown function  $u(x, \lambda)$  then satisfies a Schrödinger like equation

$$\frac{d^2 u(x, \lambda)}{dx^2} + 2w_0(x, \lambda)u(x, \lambda) - R_1 u(x, \lambda) = 0. \quad (25)$$

In order to obtain rational extensions and to connect with the exceptional polynomials, the constant  $R_1$  will be chosen so as to give polynomial solutions for  $u(x)$ . Let  $P_m(x)$  denote a polynomial solution, of degree  $m$ , of (25). A rational extension of the original SIP  $V(x)$ , will be obtained from the polynomial solutions  $P_m(x)$  for  $u(x)$  by means of the relations

$$\tilde{V}^{(-)}(x) = \tilde{w}^2(x) - \tilde{w}'(x), \quad (26)$$

where

$$\tilde{w}(x) = w_0(x) + \phi(x), \quad \phi(x) = \frac{1}{P_m(x)} \frac{dP_m(x)}{dx}. \quad (27)$$

The equations (25) - (27) are the basic equations for the rational extensions and to arrive at the Hamiltonians which have eigenfunctions in terms of the EOPs.

In general when  $R_1$  is not restricted and is allowed real values, we would get an extended potential which would interpolate between the potentials related to the exceptional polynomials.

## 4 Rational Extensions of the Radial Oscillator

### 4.1 Solutions for the Superpotential

In this section we will construct the rational extensions of the radial oscillator given by ( $2m = 1$ )

$$V(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2}. \quad (28)$$

Four superpotentials  $w(r)$  for the radial oscillator, obtained from the solutions of the QHJ equation,

$$w^2(r) - w'(r) = V(r) - E, \quad (29)$$

and the corresponding partner potentials  $V^{(\pm)}(r)$  are given in the table below.

Table 1

$k$	Superpotential $w_k$	$V_k^{(-)} = w_k^2 - w'_k$	$V_k^{(+)} = w_k^2 + w'_k$
1	$\frac{1}{2}\omega r - \frac{(\ell+1)}{r}$	$V(r) - \omega(\ell+3/2)$	$\frac{1}{4}\omega^2 r^2 + \frac{(\ell+1)(\ell+2)}{r^2} - \omega(\ell+1/2)$
2	$\frac{1}{2}\omega r + \frac{\ell}{r}$	$V(r) + \omega(\ell-1/2)$	$\frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell-1)}{r^2} + \omega(\ell+1/2)$
3	$-\frac{1}{2}\omega r - \frac{(\ell+1)}{r}$	$V(r) + \omega(\ell+3/2)$	$\frac{1}{4}\omega^2 r^2 + \frac{(\ell+1)(\ell+2)}{r^2} + \omega(\ell+1/2)$
4	$-\frac{1}{2}\omega r + \frac{\ell}{r}$	$V(r) - \omega(\ell-1/2)$	$\frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell-1)}{r^2} - \omega(\ell+1/2)$

(30)

These four solutions of the QHJ equation are meromorphic, have no moving poles and are of the form

$$w(r, \sigma) = \frac{a}{r} + \frac{1}{2}br, \quad \sigma = \{a, b\}, \quad (31)$$

where  $a, b$  have two possible values

$$a = -(\ell+1), \ell; \quad b = \pm\omega. \quad (32)$$

The four superpotentials correspond to the four possible combinations of values for  $\{a, b\}$ . The parameter values,

$$\mu = \{-\ell-1, \omega\}, \quad \lambda = \{\ell, -\omega\}. \quad (33)$$

correspond to the two solutions  $< 1 >$  and  $< 4 >$  in (30) of QHJE.

With translation mapping  $\tau$  defined as

$$\tau(\{\ell, \omega\}) = \{\ell+1, \omega\}$$

we have  $\tau(\lambda) = \{\ell+1, -\omega\}$ . The SI property (11),

$$w(r, \mu) = -w(r, \tau(\lambda)), \quad (34)$$

for the radial oscillator, is obviously satisfied. The solutions  $w_2(x), w_3(x)$  for the superpotential, in the second and the third rows of the table correspond to broken supersymmetry. These also are related by a relation of the form (34). These will be used for constructing rational potentials related to the  $L_1$  and  $L_2$  type exceptional Laguerre polynomials.

The extended potentials corresponding to  $w_1(r)$  and  $w_4(r)$  are also constructed. Most of these potentials turn out to be singular. However in some cases regular rational extensions are found and the corresponding polynomial solutions are presented in Sec 4.4.

## 4.2 Isospectral Shift Deformation

The isospectral shift for the radial oscillator will be used to construct a new potential  $\tilde{V}(r)$  by making use of Eqs. (22)-(27). Identifying  $w_0(r)$  with  $w_2(r)$ , we introduce

$$\tilde{w}_2(r) = w_2(r) + \phi(r), \quad (35)$$

and write  $\phi(r) = \frac{d}{dr} \log u(r)$ , the equation (25) then becomes

$$u''(r) + \left( \omega r + \frac{2\ell}{r} \right) u'(r) - Ru(r) = 0. \quad (36)$$

That this equation reduces to the Laguerre equation, is most easily seen by making a change of variable  $\eta = -\frac{1}{2}\omega r^2$  which transforms the above equation to

$$\eta u''(\eta) + \left( -\eta + \ell + \frac{1}{2} \right) u'(\eta) + \frac{R}{2\omega} u(\eta) = 0. \quad (37)$$

Demanding that the solution for  $u(\eta)$  be a polynomial in  $\eta$  of degree  $m$ , gives  $R = 2m\omega$  and the polynomial solution coincides with the associated Laguerre polynomials [34], [35] and we have

$$u(\eta) = L_m^\alpha(\eta), \quad \phi(r) = -\omega r \frac{\partial_\eta L_m^\alpha(\eta)}{L_m^\alpha(\eta)}, \quad \alpha = (\ell - 1/2). \quad (38)$$

Therefore, indicating the dependence of the solutions on  $m$ , the hierarchy index explicitly, we write

$$\tilde{w}_{2,m}(y) = w_2(y) + \omega r \frac{\partial_y L_m^\alpha(y)}{L_m^\alpha(y)}. \quad (39)$$

where  $y = \frac{1}{2}\omega r^2 = -\eta$  is the variable that will appear in all final expressions. Substituting (39) in (26) gives the final expression for the rational extension as

$$\tilde{V}_m^{(-)}(r) = V^{(-)}(r) + 2\omega r \left[ \omega r + \frac{2\ell}{r} \right] \frac{\partial_y L_m^\alpha(y)}{L_m^\alpha(y)} - \omega r \partial_y \left[ \omega r \frac{\partial_y L_m^\alpha(y)}{L_m^\alpha(y)} \right] + \left( \omega r \frac{\partial_y L_m^\alpha(y)}{L_m^\alpha(y)} \right)^2. \quad (40)$$

$\tilde{V}_m^{(-)}(r)$  to show the dependency on the hierarchy index  $m$ . For  $m = 0$  all the rational expressions reduce to the undeformed potentials. For each value of  $m > 0$ , we get a distinct rational extension of the radial oscillator. The partner potential  $\tilde{V}^{(+)}(r)$

$$\tilde{V}^{(+)}(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell-1)}{r^2} + 2m\omega, \quad (41)$$

by construction, coincides with the radial oscillator. The eigenfunctions of  $\tilde{V}_m^{(-)}(r)$  can be obtained by using the (broken) SUSY intertwining relation between the solutions of the partners, which for this case becomes

$$\tilde{\psi}_{n,m}^{(-)}(y) = \left( -\frac{d}{dr} + \tilde{w}_{2m}(r) \right) \tilde{\psi}_n^{(+)}(r), \quad (42)$$

where

$$\tilde{\psi}_n^{(+)}(r) = y^{\ell/2} \exp(-y/2) L_n^\alpha(y)|_{y=\frac{1}{2}\omega r^2}, \quad (43)$$

are obtained by replacing  $\ell \rightarrow \ell - 1$  in the eigenfunctions of the radial oscillator (104) in the appendix. On simplification, as explained in the appendix, (42) gives

$$\tilde{\psi}_{n,m}^{(-)}(r) = \left[ \frac{r^{\ell/2} \exp(-\frac{1}{4}\omega r^2)}{L_m^\alpha(-y)} \left( L_m^{(\alpha+1)}(-y) L_n^\alpha(y) - L_m^\alpha(-y) \partial_y L_n^\alpha(y) \right) \right]_{y=\frac{1}{2}\omega r^2}. \quad (44)$$

The recurrence relation [34]

$$\frac{d}{dr} L_n^\alpha(r) = L_n^\alpha(r) - L_n^{\alpha+1}(r) \quad (45)$$

has been used to write the eigenfunction in the form (44). The details of this step can be found in the appendix. The eigenfunctions of the extended potential have the form

$$\tilde{\psi}_{n+1,m}^{(-)}(r) = \left( \frac{r^{\ell/2} \exp(-\frac{1}{4}\omega r^2)}{L_m^\alpha(-y)|_{y=\frac{1}{2}\omega r^2}} \right) \tilde{P}_{n,m}(r), \quad (46)$$

where

$$\tilde{P}_{n,m}(r) = \left[ L_m^{(\ell+\frac{1}{2})}(-y) L_n^{(\ell-\frac{1}{2})}(y) - L_m^{(\ell-\frac{1}{2})}(-y) \partial_y L_n^{(\ell-\frac{1}{2})}(y) \right]_{y=\frac{1}{2}\omega r^2}. \quad (47)$$

The polynomials  $\tilde{P}_{n,m}(r)$  are the  $X_m$  EOPs, orthogonal with respect to the weight function

$$\mathcal{W}(r) = \exp \left( - \int \tilde{w}_{2,m}(r) dr \right) = \frac{r^{\ell/2} \exp \left( -\frac{1}{4}\omega r^2 \right)}{L_m^\alpha \left( -\frac{1}{2}\omega r^2 \right)}, \quad (48)$$

in the interval  $0 < r < \infty$ . The above polynomials have  $n + m$  zeros, where  $n$  real zeros are inside the interval of orthogonality and  $m$  zeros, either real or complex, lie outside the interval. For all the states, the number of zeros outside the orthogonality interval remains fixed, while the number of real zeros located in the interval is governed by the oscillation theorem. These polynomials in (47) reduce to 'undeformed polynomials'  $L_n^{\ell+1/2}(\omega r^2/2)$  when  $m$  is set equal to zero. From the above, one can see that each value of  $m$  gives a different rational extension of the radial oscillator (28), whose  $n^{\text{th}}$  excited state wave function can be obtained from (46). For example, putting  $m = 1$  in the equations (40) and simplifying gives

$$\tilde{V}_1^{(-)}(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2} + \frac{2\omega^2 r^2 + 4\omega\ell - 2\omega}{\omega r^2 + 2\ell + 1} + \frac{8\omega^2 r^2}{(\omega r^2 + 2\ell + 1)^2} + \left(\ell - \frac{1}{2}\right)\omega \quad (49)$$

which matches with the potential constructed by Quesne in [3]. solutions. Similarly substituting  $m = 1$  in (44), we obtain

$$\tilde{\psi}_{n,1}^{(-)}(r) = \left( \frac{r^{\ell+1} \exp \left( -\frac{1}{4}\omega r^2 \right)}{\omega r^2 + 2\ell + 1} \right) \left[ (\omega r^2 + 2\ell + 1) L_n^{(\ell+\frac{1}{2})} \left( \frac{1}{2}\omega r^2 \right) + L_n^{(\ell-\frac{1}{2})} \left( \frac{1}{2}\omega r^2 \right) \right], \quad (50)$$

where the polynomial part coincides with the  $X_1$  exceptional Laguerre polynomials of  $L_1$  series [3], [10].

### 4.3 Shape invariance of the extended potential

We arrived at the new potential  $\tilde{V}^{(-)}(r)$  by analysing the SI condition (12) for the extended potential. However, at this stage the SI of the extended potential, though guaranteed, is not transparent. This is because the SUSY partner of  $\tilde{V}^{(-)}(r)$  obtained using  $\tilde{w}_{2,m}$  is, by definition, same as potential  $V^{(+)}(r)$  apart from a constant. In order to explicitly demonstrate the SI, we obtain a new superpotential satisfying the twin properties given at the end of Sec. 2. For this purpose, we now start with  $w_3(r)$  and carry out a similar process with roles of partners exchanged. Specifically, we construct a rational extension  $\overline{w}(r) = w_3(r) + \chi(r)$  and carry out isospectral shift deformation, exchanging roles of  $V^{(\pm)}(r)$ . In this case, the solution for  $\chi(r)$  is obtained by demanding

$$\overline{V}^{(-)}(r) = V^{(-)}(r) + R_2 \quad (51)$$

and construct an extended potential

$$\overline{V}^{(+)}(r) = \overline{w}(r)^2 + \overline{w}'(r), \quad (52)$$

which turns out to be the SUSY partner of  $\tilde{V}(r)$  obtained in the previous section. Going through the same steps as above, the equation for  $\bar{w}(r)$  turns out to be

$$\chi^2(r) + 2w_3(r)\chi(r) - \chi'(r) = R_2, \quad (53)$$

where  $R_2$  is a constant depending on the potential parameters  $\lambda$  and

$$\overline{w}(r) = w_3(r) + \chi(r), \quad \chi(r) = -\frac{1}{v(r)} \frac{dv(r)}{dr}. \quad (54)$$

The equation satisfied by  $v(r)$  can be seen to Laguerre equation by changing variable to  $\eta = -\frac{1}{2}\omega r^2$ ,

$$\eta v''(\eta) - \left( -\eta - (\ell + 1) + \frac{1}{2} \right) v'(\eta) - \frac{R_2}{2\omega} v(\eta) = 0. \quad (55)$$



Again restricting to polynomial solutions and using  $y = -\eta$ , we get

$$v(y) = L_m^{\ell+1/2}(-y)|_{y=\frac{1}{2}\omega r^2}, \quad \bar{w}(r) = w_3(r) - \omega r \left[ \frac{\partial_y L_m^{(\alpha+1)}(-y)}{L_m^{(\alpha+1)}(-y)} \right]_{y=\frac{1}{2}\omega r^2}. \quad (56)$$

Substituting  $\tilde{w}(r)$  in (52) gives the expression for  $\bar{V}^{(+)}(r)$ . We will now find a new superpotential  $W_0(r)$  such that the two extended potentials  $\tilde{V}^{(-)}(r)$  and  $\bar{V}^{(+)}(r)$  can be seen to be SUSY partners, viz.,

$$\tilde{V}^{(-)}(r) = W_0^2(r) - W_0'(r) \quad (57)$$

$$\text{and } \bar{V}^{(+)}(r) = W_0^2(r) + W_0'(r). \quad (58)$$

For this we collect all the equations relating the potentials  $\tilde{V}^{(\pm)}(r)$  and  $\bar{V}^{(\pm)}(r)$  and list them below.

$$\tilde{V}^{(+)}(r) = V^{(+)}(r) + R_1, \quad (59)$$

$$\bar{V}^{(-)}(r) = V^{(-)}(r) + R_2, \quad (60)$$

$$\tilde{V}^{(-)}(r) = \tilde{w}(r)^2 - \tilde{w}'(r), \quad (61)$$

$$\bar{V}^{(+)}(r) = \bar{w}(r)^2 + \bar{w}'(r). \quad (62)$$

Note that (59) implies

$$\phi^2(r, \lambda) + 2w_2(r, \lambda)\phi(r, \lambda) = R_1 - \phi'(r, \lambda). \quad (63)$$

Similarly, the equation for (60) implies that  $\chi(r)$  obeys a relation

$$\chi^2(r, \lambda) - 2w_3(r, \lambda)\chi(r, \lambda) = R_2 + \chi'(r, \lambda). \quad (64)$$

Using (57) in the l.h.s of (61) and substituting  $W_0(r) = w_1(r, \lambda) + \xi(r, \lambda)$  gives  $\tilde{w}(r) = w_3(r) + \phi(r)$  to get

$$\text{L.H.S. of (61)} = (w_1(r, \lambda) + \xi(r, \lambda))^2 - (w_1'(r, \lambda) + \xi'(r, \lambda)) \quad (65)$$

$$\begin{aligned} &= w_1^2(r, \lambda) - w_1'(r, \lambda) + \xi(r, \lambda)^2 + 2w_1(r, \lambda)\xi(r, \lambda) - \xi'(r, \lambda) \\ &= V(r) - \omega(\ell + 3/2) + \xi(r, \lambda)^2 + 2w_1(r, \lambda)\xi(r, \lambda) - \xi'(r, \lambda). \end{aligned} \quad (66)$$

Next, the r.h.s of (61) is expressed in terms of  $\tilde{w}(r) = w_2(r) + \phi(r)$  gives

$$\text{R.H.S. of (61)} = (w_2(r, \lambda) + \phi(r, \lambda))^2 - (w_2'(r, \lambda) + \phi'(r, \lambda)) \quad (67)$$

$$\begin{aligned} &= w_2^2(r, \lambda) - w_2'(r, \lambda) + \phi^2(r, \lambda) + 2w_2(r, \lambda)\phi(r, \lambda) - \phi'(r, \lambda) \\ &= V(r) + \omega(\ell - 1/2) + \phi^2(r, \lambda) + 2w_2(r, \lambda)\phi(r, \lambda) - \phi'(r, \lambda). \end{aligned} \quad (68)$$

Equating the expressions in (66) and (68) we get

$$\xi^2(r, \lambda) + 2w_1(r, \lambda)\xi(r, \lambda) - \xi'(r, \lambda) \quad (69)$$

$$= 2\omega(\ell + 1) + \phi^2(r, \lambda) + 2w_2(r, \lambda)\phi(r, \lambda) - \phi'(r, \lambda) \quad (70)$$

$$= 2\omega(\ell + 1) + R_1 - 2\phi'(r, \lambda), \quad (71)$$

where, in the last step, (63) has been used. Similarly from (59) and (64) we would get

$$\xi^2(r, \lambda) + 2w_1(r, \lambda)\xi(r, \lambda) + \xi'(r, \lambda) \quad (72)$$

$$= 2\omega(\ell + 1) + \chi^2(r, \lambda) + 2w_2(r, \lambda)\chi(r, \lambda) - \chi'(r, \lambda) \quad (73)$$

$$= 2\omega(\ell + 1) + R_2 - 2\chi'(r, \lambda). \quad (74)$$

Subtracting (71) and (74) we get

$$\xi'(r, \lambda) = \phi'(r, \lambda) - \chi'(r, \lambda) + A, \quad (75)$$

where  $A$  is a constant to be fixed. Integrating we get

$$\xi(r, \lambda) = \phi(r, \lambda) - \chi(r, \lambda) + Ar + B. \quad (76)$$

Both the constants  $A$  and  $B$  turn out to be zero. This is seen by noting that the large  $r$  behaviour of the potential (40) is correctly reproduced by  $w_1(r, \lambda)$  term in  $W_0(r) = w_1(r, \lambda) + \xi(r, \lambda)$ . Therefore any addition of  $Ar + B$  with  $A, B \neq 0$  in  $W_0(r)$  is not possible, Thus we arrive at the result

$$W_0(r, \lambda) = w_1(r, \lambda) + \phi(r, \lambda) - \chi(r, \lambda). \quad (77)$$

$$= \frac{1}{2}\omega r - \frac{(\ell+1)}{r} - \omega r \left[ \frac{\partial_y L_m^\alpha(-y)}{L_m^\alpha(-y)} \right]_{y=\frac{1}{2}\omega r^2} + \omega r \left[ \frac{\partial_y L_m^{(\alpha+1)}(-y)}{L_m^{(\alpha+1)}(-y)} \right]_{y=\frac{1}{2}\omega r^2}. \quad (78)$$

In this form the superpotential for the extended radial oscillator agrees with that known in literature [3], [10].

The expression for  $W_0(r, \lambda)$  can also be obtained by an alternate approach by noting that  $W_0(r, \lambda)$  is just the superpotential constructed out of the ground state wave function of the potential  $\tilde{V}^{(-)}(r)$ . The wave functions for  $\tilde{V}^{(-)}(r)$  are given by (42). Taking logarithmic derivative of the ground state wave function gives an expression for  $W_0(r, \lambda)$  that matches with (78). The details of this computation are given in the appendix.

The process of construction of the rational extension of the radial oscillator potential described in detail above leads to extended Hamiltonian related to the  $L_1$  exceptional orthogonal Laguerre polynomials. The above process of rational extension of potentials by isospectral shift deformation is summarized in the the left half of the flowchart given in figure 1. In addition, we also give the steps to construct its shape invariant partner in the right half.

#### 4.4 $L_2$ and $L_3$ rational extensions

In this section we will briefly summarize our results on rationally extended potentials obtained when a different route to the rational extension is taken.

##### **$L_2$ Category**

First we would like to mention that the entire process leading to the exceptional Laguerre polynomials of  $L_1$  category can be repeated by exchanging the roles of  $w_2(r)$  and  $w_3(r)$ . The two extended potentials, obtained by these procedures, correspond to the SUSY partner Hamiltonians listed as being related to exceptional Laguerre polynomials of  $L_2$  category by Otake and Sasaki [10].

Here, we give the final expressions for the  $L_2$  exceptional Laguerre polynomials

$$\tilde{P}_{n,m} = \frac{1}{(\ell - m + 1/2)} \left[ (\ell - m + 1/2) L_m^{-\ell-3/2}(y) L_n^{\ell+1/2}(y) + y L_m^{-\ell-1/2}(y) \partial_y L_n^{\ell+1/2}(y) \right]_{y=\frac{1}{2}\omega r^2}, \quad (79)$$

The corresponding weight function is given by

$$\mathcal{W}_m = \frac{r^\ell \exp(-\omega r^2/4)}{L_m^{-\ell-1/2}(y)|_{\frac{1}{2}\omega r^2}}. \quad (80)$$

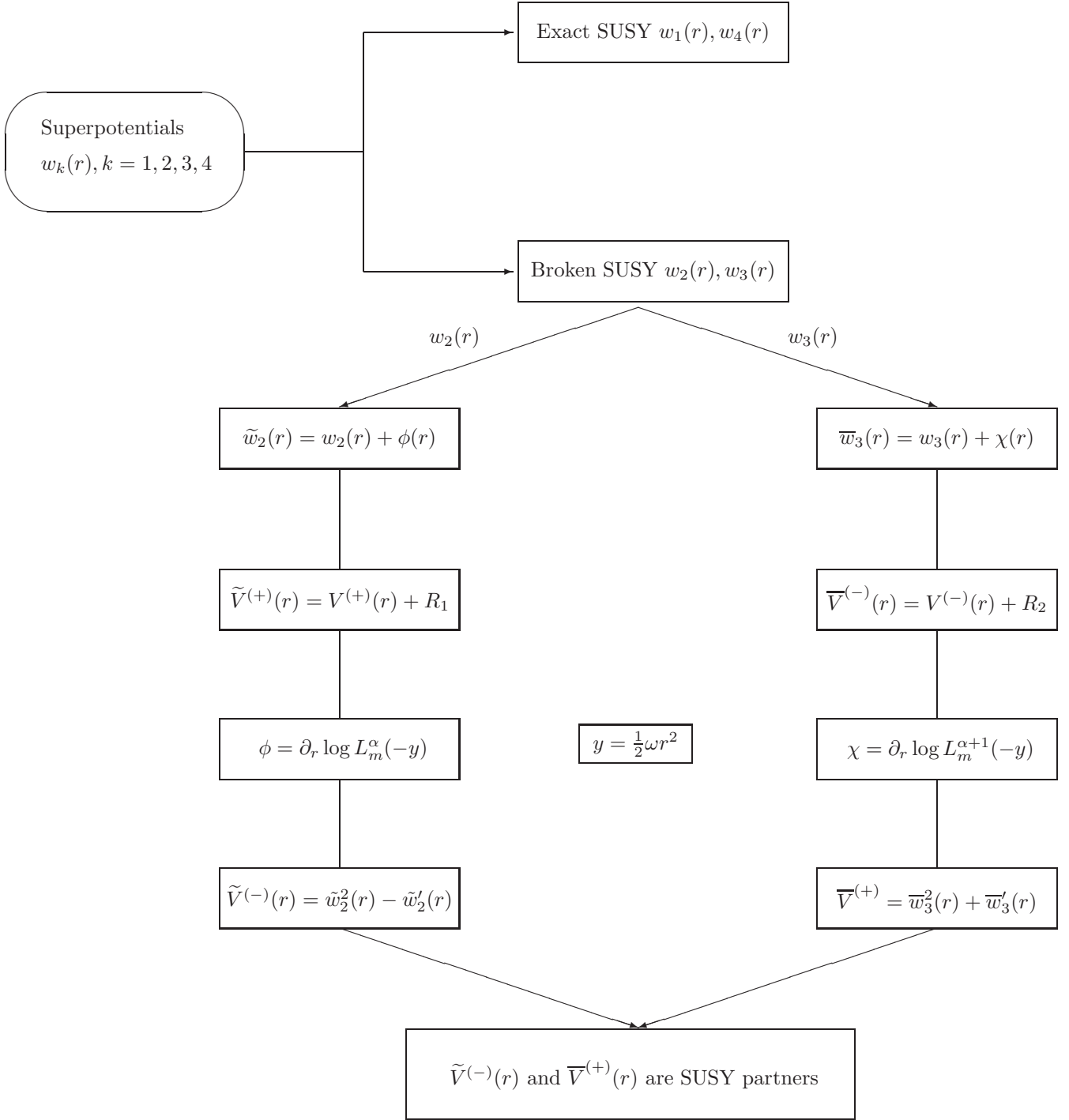


Fig.1.

### L3 Category

The method of spectral shift deformation when applied to the superpotentials  $w_2(r)$  and  $w_3(r)$  yielded rational extensions of the radial oscillator potential. These potentials were shown to have solutions in terms of the  $L1$  and  $L2$   $X_m$  exceptional Laguerre polynomials. The above process can be repeated with the superpotentials  $w_1(r)$  and  $w_4(r)$  and we can construct further rational extensions of the radial oscillator. From the table it is clear that  $w_4(r) = -w_1(r, (l+1) \rightarrow l)$ . It is seen that the isospectral shift deformation, using these two superpotentials leads to two new rational extensions of which one is a singular potential and not interesting in the present context. We look at the other potential which is non-singular for certain values of the corresponding parameters. In general, it has singularities both in the complex plane and the physical region. These singularities are governed by the zeros of the polynomial in the corresponding  $\phi(r)$ . The explicit form of the new generalized rational extension and its polynomial solutions are presented here for the first time.

The rational extension obtained by performing the isospectral shift deformation using  $w_1(r)$  and demanding  $\tilde{V}_1^{(+)}(r) = V_1^{(+)}(r) + R_3$  gives

$$\tilde{V}_m^{(-)}(r) = \left[ V^{(-)}(r) + 2\omega r \left( \frac{\omega r}{2} - \frac{\ell+1}{2} \right) \frac{\partial_y L_m^\alpha(-y)}{L_m^\alpha(-y)} - \omega r \partial_y \left( \omega r \frac{\partial_y L_m^\alpha(-y)}{L_m^\alpha(-y)} \right) + 2 \left( \omega r \frac{\partial_y L_m^\alpha(-y)}{L_m^\alpha(-y)} \right)^2 \right]_{y=\frac{1}{2}\omega r^2}, \quad (81)$$

where  $\alpha = -l - 3/2$  and again  $m = 1, 2, 3, \dots$ . The corresponding eigenfunctions are given by

$$\tilde{\psi}_{n,m}^-(r) = \left( \frac{y^{(l+2)/2} \exp(-\frac{1}{4}\omega r^2)}{L_m^\alpha(\frac{1}{2}\omega r^2)} \right) \tilde{P}_{n,m}(r), \quad (82)$$

where

$$\tilde{P}_{n,m}(r) = \left[ \omega r L_n^{-\alpha+1}(y) L_m^\alpha(-y) + \frac{2}{r} (m - l - \frac{3}{2}) L_m^{\alpha-1}(-y) L_n^\alpha(y) \right]_{y=\frac{1}{2}\omega r^2}, \quad (83)$$

are polynomials orthogonal with respect to the weight function

$$\mathcal{W}_m(r) = \frac{r^{\ell+2} \exp(-\omega r^2/4)}{L_m^\alpha(-y)|_{y=\frac{1}{2}\omega r^2}} \quad (84)$$

in the interval  $0 < r < \infty$ . The weight function has singularities governed by the zeros of the polynomial  $L_m^\alpha(-\omega r^2/2)$ . From the Kienast-Lawton-Hahn's theorem on the zeros of the Laguerre polynomials [36], [37], the polynomial  $L_m^\alpha(-\omega r^2/2)$  will not have any zeros on the positive real line as the argument of the polynomial is strictly negative. But the polynomial will have one negative zero if  $m$  is odd and  $\alpha$  lies in the range  $(-m - \frac{1}{2}) < \alpha < -m$ . Thus for even  $m$  and  $\alpha$  not lying in the above given range, the weight function is non-singular in the physical range. Thus for all such values of  $m$  and  $\alpha$ ,  $\tilde{V}^{(-)}(r)$  has polynomial solutions. From the above discussion it is clear that the polynomials  $\tilde{P}_{n,m}(r)$  are orthogonal and form a complete set. In addition, the polynomials start with a degree  $m > 0$ . Thus we have a new set of exceptional polynomials and following the nomenclature in the literature, we name them the  $L3$  exceptional polynomials. The discussion of the properties of this new series of the polynomials falls outside the domain of this paper and will be taken up elsewhere.

Thus the isospectral shift deformation using  $w_1(r)$  leads to a rational extension of the radial oscillator, whose solutions are in terms of new set of exceptional polynomials named as the  $L3$  series. Rational extensions of radial oscillator having solutions other than  $L1$  and  $L2$  Laguerre EOPs have been discussed in [8] and [13].

## 5 Trigonometric Pöschl-Teller Potential

The trigonometric Pöschl-Teller potential is given by

$$V(x) = A(A+1)\text{cosec}^2 x + B(B+1)\sec^2 x, \quad A > -\frac{1}{2}, B > -\frac{1}{2}. \quad (85)$$

The form of the superpotentials is given by

$$w(x, \sigma) = a \cot x - b \tan x \quad (86)$$

with  $\sigma$  collectively denoting the parameters  $\{a, b\}$ . The possible values of the parameters are

$$a = A, -A - 1, \quad b = B, -B - 1 \quad (87)$$

and the four solutions of the QHJE correspond to the following choices

$$\begin{aligned} \langle 1 \rangle \quad E &= -(A + B)^2, & \sigma &= \{A, B\} \\ \langle 2 \rangle \quad E &= -(1 + A - B)^2, & \sigma &= \{-A - 1, B\} \\ \langle 3 \rangle \quad E &= -(1 - A + B)^2, & \sigma &= \{A, -B - 1\} \\ \langle 4 \rangle \quad E &= -(2 + A + B)^2, & \sigma &= \{-A - 1, -B - 1\} \end{aligned} \quad (88)$$

The corresponding solutions will be denoted as  $w_k, k = 1, 2, 3, 4$ , respectively. With the mapping  $\tau$  defined as

$$\tau\{A, B\} = \{A + 1, B + 1\},$$

the shape invariance property is obvious from  $w_4(x, \lambda) = -w_1(x, \tau(\lambda))$  and from  $w_3(x, \lambda) = -w_2(x, \tau(\lambda))$ .

## 5.1 Rational Extension of Trigonometric Pöschl-Teller Potential

We note that the first solution in (88) corresponds to the exact SUSY case and to a normalizable ground state solution. Solutions  $\langle 2 \rangle$  and  $\langle 3 \rangle$  in (88) correspond to broken SUSY and will be used to construct rational extensions with solutions corresponding to  $J1$  and  $J2$  exceptional Jacobi polynomials [3], [8], [10].

To keep our intermediate steps general we take

$$w(x, \sigma) = a \cot x - b \tan x \quad (89)$$

and will substitute suitable values for the constants  $\sigma = \{a, b\}$  at the end. The first extension process is as follows.

$$\tilde{V}^{(\pm)} = \tilde{w}(x)^2 \pm \tilde{w}'(x) \quad (90)$$

$$\tilde{w}(x) = w(x, \sigma) + \phi(x), \quad (91)$$

where  $\tilde{w}(x)$  is determined by demanding

$$\tilde{V}^{(+)}(x) = V^{(+)}(x) + K, \quad (92)$$

where  $K$  is a constant.  $\phi(x)$  satisfies the Riccati equation

$$\phi^2(x) + 2w(x)\phi(x) + \phi'(x) = K. \quad (93)$$

This equation can be linearised by writing

$$\phi(x) = \frac{u'(x)}{u(x)} \quad (94)$$

and the equation for  $u(x)$  turns out to be

$$u'' + 2w(x, \sigma)u' - Ku = 0 \quad (95)$$

$$u'' + 2(a \cot x - b \tan x)u' - Ku = 0. \quad (96)$$

A point canonical transformation  $y = \cos 2x$  leads to the equation

$$(1 - y^2) \frac{d^2 u}{dy^2} + [b - a + (a + b + 1)y] \frac{du}{dy} - K_1 u = 0 \quad (97)$$

where  $K_1 = K/4$  is a constant. This equation has polynomial solutions with  $K_1 = N(N + \nu + \mu + 1)$  with  $\{\nu, \mu\} = \{a - 1/2, b - 1/2\}$ . With this choice of  $K_1$  the polynomial solution coincides with the Jacobi polynomial  $P_N^{(\nu, \mu)}(y)$ . Therefore, we get

$$\phi(x, \lambda) = \frac{1}{u} \left( \frac{dy}{dx} \right) \left( \frac{du(y, \lambda)}{dy} \right) \quad (98)$$

$$= -(\sin 2x) \left[ \frac{d}{dy} \log P_N^{(a-\frac{1}{2}, b-\frac{1}{2})}(y) \right]_{y=\cos 2x}. \quad (99)$$

One can obtain other rationally extended potentials  $\bar{V}^{(+)}(x)$  by means of a second extension process, as done in the case of radial oscillator. This is done by writing

$$\bar{w}(x) = w(x, \sigma) + \chi(x) \quad (100)$$

and fixing  $\chi(x)$  by demanding  $\bar{V}^{(-)}(x) = V^{(-)}(x) + \bar{K}$ , which leads to the following equations for  $\chi(x)$

$$\chi(x) = -\frac{\bar{u}'(x)}{u(x)} \bar{u}'' - 2(a \cot x - b \tan x) \bar{u}' - \bar{K} \bar{u} = 0. \quad (101)$$

Here again  $\sigma = \{a, b\}$  and can take one of the four possible values listed in (88).

As in the case of radial oscillator, the two superpotentials  $w_2(x)$ ,  $w_3(x)$  correspond to broken SUSY and will give rise to two Shape invariant nonsingular extended potentials and their supersymmetric partners. We obtain two more rational extensions and their partners using the other two superpotentials  $w_1(x)$ ,  $w_4(x)$  and some of these are likely to be singular. Thus starting with the four superpotentials  $w_k(x)$ ,  $k = 1, 2, 3, 4$ , in all, there are eight possible extensions obtained by the two processes outlined above and their SI is guaranteed by the ansatz that led to equations for the isospectral shift deformation.

The rational extensions and their partners, obtained from  $w_2(x)$  and  $w_3(x)$  will correspond to the Hamiltonians related to  $J1$ , and  $J2$  exceptional Jacobi polynomials [3], [10]. Of the remaining four extensions, some will be singular potentials and some others will be regular potentials. A detailed study is needed to make a definite conclusive statement whether other regular extended potentials are related to the exceptional Jacobi polynomials already found [14] or as in the case of radial oscillator, will lead to new exceptional Jacobi polynomials. Further investigation in this direction, along with some other related studies is in progress and will be reported elsewhere.

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## 6 Concluding Remarks

In this paper we have discussed a procedure to systematically construct rational extensions of SIPs. The extended potentials are designed to be shape invariant. **Our treatment makes it possible to construct an extension which interpolates between various potentials related to  $X_m$ - exceptional polynomials for different values of  $m$ . So for example, a solution of (37) with arbitrary value of the constant  $R$  will give rise to such an extended potential, which for special values of  $R$  will reduce to potentials related to  $X_m$  exceptional polynomials for various  $m$  values.**

The method can be used to construct rational extensions of all known SIPs [5]. Rational extensions of the radial oscillator and the trigonometric DPT have been constructed explicitly. In the case of radial oscillator, we obtained the well studied rational extensions with  $L1$  and  $L2$  exceptional Laguerre polynomials as solutions. In addition, we have presented another set of infinite generalized rational potentials whose solutions involve a new set of exceptional Laguerre polynomials, whose generalized expressions are given for the first time.

In the case of the DPT potential, we obtain the extended superpotentials from which rational extensions of DPT can be constructed. It is clear that this method leads to new rational extensions other than those discussed in literature, hence further study is required to see if these lead to further new types of exceptional Jacobi polynomials. It would be interesting to investigate further generalizations of our method and see where they lead to.

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## APPENDIX

In this appendix we will give details of (44) and an alternate derivation of (78).

The superpotential and the the Hamiltonian for the radial oscillator are given by

$$w(r) = \frac{1}{2}\omega r - \frac{\ell+1}{r} \quad (102)$$

$$H^{(-)} = p^2 + \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2} - (\ell + \frac{3}{2})\omega. \quad (103)$$

The energy eigenvalues and the eigenfunctions are

$$E_n = 2n\omega; \quad \psi_n^{(-)}(y) = Ny^{(\ell+1)/2} \exp(-y/2) L_n^{\ell+1/2}(y), \quad (104)$$

where  $y = \frac{1}{2}\omega r^2$ .

The eigenfunctions for  $H^{(+)}(r)$ , as obtained from the above expressions, by replacing  $\ell \rightarrow \ell - 1$  are

$$\psi_n^{(+)}(y) = Ny^{\ell/2} \exp(-y/2) L_n^{\alpha}(y), \quad \alpha = (\ell - 1/2). \quad (105)$$

A rational extension was obtained by starting from  $w_2 = \frac{1}{2}\omega r + \ell/r$  and introducing  $\tilde{w}(r) = w_2(r) + \phi(r)$ . We demanded that  $\phi(r)$  satisfy  $\tilde{V}^{(+)}(r) = V^{(+)}(r) + R_1$  and be a rational function of  $r$ . This led to

$$\tilde{w}_{2,m}(r) = w_2(r) + \omega r \left( \frac{\partial_y L_m^{\alpha}(-y)}{L_m^{\alpha}(-y)} \right). \quad (106)$$

The eigenfunctions of the extended potential  $\tilde{V}^{(-)}(r)$  can be computed using the intertwining relation

$$\tilde{\psi}_{n,m}^{(-)}(r) = \left( -\frac{d}{dr} + \tilde{w}_{2,m}(r) \right) \tilde{\psi}_n^{(+)}(r). \quad (107)$$

For this purpose first note that  $\tilde{\psi}_n^{(+)}(r) = \psi_n^{(+)}(r)$  of (105), which leads to

$$\frac{d}{dr} \tilde{\psi}_{n,m}^{(+)}(r) = \psi_n^{(+)}(r) \left( \frac{d}{dr} \log \psi_n^{(+)}(r) \right) \quad (108)$$

$$= \psi_n^{(+)}(y) \left( \frac{dy}{dr} \right) \left( \frac{d}{dy} \log \psi_n^{(+)}(y) \right) \quad (109)$$

$$= \psi_n^{(+)}(y) \times \omega r \left[ \frac{\ell}{2y} - \frac{1}{2} + \frac{\partial_y L_n^{\alpha}(y)}{L_n^{\alpha}(y)} \right]. \quad (110)$$

Use of the above result in (107) gives the following expression for  $\psi_{n,m}^{(-)}(r)$

$$\psi_{n,m}^{(-)}(r) = \psi_n^{+}(r) (\omega r) \left[ -\frac{\ell}{2y} + \frac{1}{2} - \frac{\partial_y L_n^{\alpha}(y)}{L_n^{\alpha}(y)} \right] \quad (111)$$

$$+ \frac{1}{2} + \frac{\ell}{2y} + \frac{\partial_y L_m^{(\alpha)}(-y)}{L_m^{\alpha}(-y)} \quad (112)$$

$$= \psi_n^{+}(r) (\omega r) \left[ \frac{L_m^{\alpha}(-y)}{L_m^{\alpha}(-y)} - \frac{\partial_y L_n^{\alpha}(y)}{L_n^{\alpha}(y)} + 1 \right]. \quad (113)$$

Using the identity

$$\partial_x L_m^{\alpha}(x) = L_m^{\alpha}(x) - L_m^{(\alpha+1)}(x), \quad (114)$$

we get

$$\tilde{\psi}_{n,m}^{(-)}(r) = (\omega r) \left( \frac{L_m^{\alpha+1}(-y)}{L_m^{\alpha}(-y)} - \frac{\partial_y L_n^{\alpha}(y)}{L_n^{\alpha}(y)} \right)_{y=\frac{1}{2}\omega r^2} \tilde{\psi}_n(r). \quad (115)$$

The final expression for the wave function  $\tilde{\psi}_{n,m}^{(-)}(r)$ , as obtained by using the equations (113) and (115) is

$$\tilde{\psi}_{n,m}^{(-)}(r) = r^{\ell/2} \exp\left(-\frac{1}{4}\omega r^2\right) \left[ \frac{L_m^{\alpha+1}(-y)L_n^{\alpha}(y) - \partial_y L_n^{\alpha}(y)L_m^{\alpha}(-y)}{L_m^{\alpha}(-y)} \right]_{y=\frac{1}{2}\omega r^2} \quad (116)$$

where  $\alpha = \ell - 1/2$ . This is the result in (44) and the expression for the eigenfunctions can be used to have an alternate derivation of (78).

Setting  $n = 0$  in (115) gives

$$\tilde{\psi}_{0,m}^{(-)}(r) = (\omega r) \left( \frac{L_m^{\alpha+1}(-y)}{L_m^{\alpha}(-y)} \right)_{y=\frac{1}{2}\omega r^2} \tilde{\psi}_{0,m}^{+}(r). \quad (117)$$

The desired superpotential can be obtained as

$$W_0(r) = -\frac{d}{dr} \left( \log \tilde{\psi}_{0,m}^{(-)}(r) \right). \quad (118)$$

Substituting the expression in (117) in the above equation gives

$$W_0(r) = -\frac{d}{dr} \left( \log \tilde{\psi}_{n,m}^{(-)}(r) \right) \Big|_{n=0} \quad (119)$$

$$= -\frac{1}{r} - \frac{d}{dr} \log \tilde{\psi}_0^{(+)}(r) + \omega r \left[ \frac{\partial_y L_m^{\alpha}(-y)}{L_m^{\alpha}(-y)} - \frac{\partial_y L_m^{(\alpha+1)}(-y)}{L_m^{(\alpha+1)}(-y)} \right]_{y=\frac{1}{2}\omega r^2}. \quad (120)$$

$$= \frac{1}{2}\omega r - \frac{\ell+1}{r} + \omega r \partial_y \log \left( \frac{L_m^{\alpha}(-y)}{L_m^{\alpha+1}(-y)} \right)_{y=\frac{1}{2}\omega r^2} - . \quad (121)$$

which agrees with (78).

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